

ASYMMETRICAL PSEUDOELASTICITY**V.O. Bytev**, Tyumen State University

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Term "asymmetrical pseudoelasticity" refers to the theory, in which a symmetrical stress tensor and a symmetrical strain tensor are connected by means of an asymmetrical material tensor. An 6-dimensional asymmetrical matrix of elasticity has been constructed that is invariable in relation to orthogonal transformation with a single rotation operator and coordinated with conservation laws of the continuum mechanics. The matrix has got eight independent components and expands the traditional definition of transversally isotropic (hexagonal) material symmetry. The suggested theory includes definition of 3-dimensional and 2-dimensional linear boundary value problems and accurate solutions generalizing the traditional polynomial solutions, A.Love's solutions, and N.I.Muskhelishvili's solutions and providing new kinematic effects.

Key words: constitutive relations, rotational invariance, asymmetrical pseudoelasticity, elasticity tensor, boundary value problems.

Introduction. The problem of detecting the constitutive relations within the continuum mechanics was set in the works of Truesdell's school [1] and the idea of applying continuous groups theory for its solution was "in the air" as back in 1912-1914 A.Einstein pointed out the necessity to look for invariants of continuous transformations groups as one of major problems of the physics. The modern stage of systematic application of group analysis methods for models of the continuum mechanics was developed in the works of L.V.Ovsiannikov's [2] and N.Kh.Ibragimov's schools. Group features of the differential equations of flow media and gas media mechanics are studied in detail at present.

Invariance principle forms the basis for modern approaches connected with searching for new types of physical structures. Within the continuum mechanics it provides the general approach to composing defining relations that are required for closure of differential equation sets resulting from integral conservation laws.

The method of solving problems of synthesis and analysis of constitutive equations within the continuum mechanics basing on the invariance principle was suggested in O.V.Bytev's works [4-6]. Differential equation set of purely mechanical non-polar continuum

$$\mathbf{v}_t + (\mathbf{v} \cdot \nabla)\mathbf{v} - \rho^{-1} \operatorname{div} \mathbf{T} + \rho^{-1} \nabla p = \mathbf{0} ,$$

$$p_t + (\mathbf{v} \cdot \nabla)p + G \operatorname{div} \mathbf{v} + H \mathbf{T} : \nabla \mathbf{v} = 0 ,$$

$$\rho_t + \operatorname{div}(\rho \mathbf{v}) = 0 , \quad G = G(p, \rho) , \quad H = H(p, \rho) , \quad \mathbf{T} = \mathbf{T}(\nabla \mathbf{v}) ,$$

resulting from conservation laws and traditional two-parameter thermodynamics were subject to group analysis with the purpose of using a group of continuous transformations for formation of the invariance principle itself (symbols used in the equations: \mathbf{v} is the vector velocity field, \mathbf{T} is the tensor field of dissipative stress, p is the balance pressure, ρ is the continuum mass density, G and H are arbitrary functions of state parameters).

Group analysis performed in [5, 6] detected that transformation of initial equations equivalency does not include SO_3 group. Thus, a full classification problem was solved without any additional suggestions (isotropy, coaxiality etc). The procedure of group classification detected the following possibilities: 1) if \mathbf{T} , G and H are arbitrary functions of their arguments, the initial system allows only Hamilton's group that forms the group germ; 2) on assumption of three rotations (spherical invariance), there are traditional dependencies between \mathbf{T} and $\nabla \mathbf{v}$; 3) on assumption of one rotation (rotational invariance), dependencies between the continuum state parameters can differ from traditional ones.

The hierarchy of constitutive relations synthesized in [6] allows to formulate non-traditional closed models of liquid and solid media. Thus, analysis of the linear dependency between a symmetrical tensor of viscous stresses and a symmetrical tensor of strain rates in case when a medium model assumes

only one rotation operator proved that these tensors can be connected by an asymmetrical transformation tensor.

On the basis of results obtained in [5, 6], some comments shall be made in respect of the traditional approach to formation of models within the continuum mechanics.

1. Theoretical approach (that has already become traditional) used to determine the stress tensor dependency of strain tensor (of strain rates tensor) is based on a postulate of stress tensor invariance in relation to actions of SO_3 group [1]. It results in symmetrical models of continuous media being in subordination to coaxiality of adjoint tensors, whereas, for instance, in ground models it is impossible to preserve coaxiality of stress and strain rate tensors. It is thus worth mentioning that a stress tensor is not an abstract tensor object, it complies with the momentum conservation law. Here the question arises: is the momentum conservation law always invariant in relation to SO_3 group action?

2. Bringing two tensor fields of the second rank to linear non-vector proportionality is equivalent to bringing two quadratic forms to a canonical form by means of a single orthogonal transformation. It is possible only if a pair of nondegenerate quadratic forms uses a regular pencil, i.e. when both coefficient matrices are symmetrical and one of the associated quadratic forms is positively definite (Sylvester's criterion) [7]. Stress tensors and deformation tensors are symmetric by definition of a simple non-polar continuum, whereas the Sylvester's criterion shall be checked.

3. The mechanics of a deformed solid body distinguishes the Cauchy elasticity when a stress tensor is defined as an invariant function of a deformation tensor and Green elasticity when elastic energy potential is postulated. According to C.Truesdell, a material with Green elasticity is referred to as hyperelastic. The procedure of composing defining equations with application of elastic potential being the quadratic form of deformation tensor components that is usual for invariant hyperelasticity theory excludes the asymmetry of elasticity tensor.

The foregoing comments formed the basis for a more detailed analysis of possible variants of defining relations within Cauchy's linear elasticity theory.

1.3-Dimensional Model of the Asymmetrical Pseudoelasticity

Let us analyze a 3-dimensional linear model of elastic medium with asymmetric elasticity tensor (of rigidity or compliance) in the Cartesian reference system x_1, x_2, x_3 . Let σ_{ij} be components of a symmetrical stress tensor, ε_{ij} – components of a symmetrical deformation tensor ($i, j = 1, 2, 3$). Let us introduce the following symbols:

$$\begin{aligned}\sigma_1 &= \sigma_{11}, \sigma_2 = \sigma_{22}, \sigma_3 = \sigma_{12} = \sigma_{21}, \\ \sigma_4 &= \sigma_{31} = \sigma_{13}, \sigma_5 = \sigma_{32} = \sigma_{23}, \sigma_6 = \sigma_{33}, \\ \varepsilon_1 &= \varepsilon_{11}, \varepsilon_2 = \varepsilon_{22}, \varepsilon_3 = 2\varepsilon_{12} = 2\varepsilon_{21}, \\ \varepsilon_4 &= 2\varepsilon_{31} = 2\varepsilon_{13}, \varepsilon_5 = 2\varepsilon_{32} = 2\varepsilon_{23}, \varepsilon_6 = \varepsilon_{33}\end{aligned}$$

and column-vectors

$$\sigma = [\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5, \sigma_6]^T, \quad \varepsilon = [\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5, \varepsilon_6]^T.$$

The most general linear connection between Cartesian tensors σ_{ij} and ε_{ij} is expressed by the constitutive relation $\sigma_M = C_{MN}\varepsilon_N$, where capital indices range over values 1 – 6, thus C_{MN} coefficients form the elasticity matrix of the 6th rank. Let us perform an orthogonal transformation with a single rotation operator with the analyzed elasticity relation and define conditions of this relation being invariable in relation to this transformation. In order to do so, we shall perform a right-hand rotation of axes in relation to x_3 axis:

$$x'_1 = x_1 \cos \varphi + x_2 \sin \varphi, \quad x'_2 = -x_1 \sin \varphi + x_2 \cos \varphi, \quad x'_3 = x_3,$$

where φ is a random angle. Strain tensor (and stress tensor) components are transformed according to the rule:

$$\begin{aligned}\varepsilon'_{11} &= \varepsilon_{11} \cos^2 \varphi + \varepsilon_{22} \sin^2 \varphi + 2\varepsilon_{12} \cos \varphi \sin \varphi, \\ \varepsilon'_{22} &= \varepsilon_{11} \sin^2 \varphi + \varepsilon_{22} \cos^2 \varphi - 2\varepsilon_{12} \cos \varphi \sin \varphi, \\ \varepsilon'_{12} &= \varepsilon_{12}(\cos^2 \varphi - \sin^2 \varphi) - (\varepsilon_{11} - \varepsilon_{22}) \cos \varphi \sin \varphi, \\ \varepsilon'_{31} &= \varepsilon_{31} \cos \varphi + \varepsilon_{32} \sin \varphi, \\ \varepsilon'_{32} &= -\varepsilon_{31} \sin \varphi + \varepsilon_{32} \cos \varphi, \quad \varepsilon'_{33} = \varepsilon_{33}.\end{aligned}$$

The condition of rotational invariance of elasticity relations is expressed by formula $\sigma'_M = C_{MN}\varepsilon'_N$ considering the rule of tensors transformation.

Thus, relation $\sigma'_6 = C_{6N}\varepsilon'_N$ requires the following equalities being fulfilled

$$C_{62} = C_{61}, \quad C_{63} = C_{64} = C_{65} = 0,$$

relation $\sigma'_4 = C_{4N}\varepsilon'_N$ requires fulfillment of equalities

$$C_{41} = C_{42} = C_{43} = C_{46} = 0, \\ C_{51} = C_{52} = C_{53} = C_{56} = 0, \quad C_{54} = -C_{45}, \quad C_{55} = C_{44}$$

and relation $\sigma'_2 = C_{2N}\varepsilon'_N$ requires fulfillment of equalities

$$C_{14} = C_{15} = C_{24} = C_{25} = C_{34} = C_{35} = C_{36} = 0, \\ C_{21} = C_{12}, \quad C_{22} = C_{11}, \quad C_{23} = -C_{13}, \quad C_{26} = C_{16}, \\ C_{31} = -C_{13}, \quad C_{32} = C_{13}, \quad 2C_{33} = C_{11} - C_{12}.$$

The other of equations do not provide any additional restrictions.

Ultimately, the system of constitutive relations for a linear model with a single rotation operator can be represented in a matrix form

$$\sigma = C\varepsilon, \quad C = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & C_{16} \\ C_{12} & C_{11} & -C_{13} & 0 & 0 & C_{16} \\ -C_{13} & C_{13} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & C_{45} & 0 \\ 0 & 0 & 0 & -C_{45} & C_{44} & 0 \\ C_{61} & C_{61} & 0 & 0 & 0 & C_{66} \end{bmatrix}, \quad (1.1)$$

where C is elasticity matrix 6×6 . There is equality $C_{12} = C_{11} - 2C_{33}$, the remaining 8 components: C_{11} , C_{13} , C_{16} , C_{33} , C_{44} , C_{45} , C_{61} , C_{66} are independent kinetic parameters, moreover C_{13} and C_{45} can be either positive or negative.

Elasticity matrix present in (1.1) is different from a traditional matrix of material rigidity with transversally isotropic (hexagonal) symmetry by presence of asymmetric components and transforms into traditional one on conditions $C_{45} = C_{13} = 0$, $C_{61} = C_{16}$.

Sylvester's criterion confirms that relation (1.1) ensures positive definiteness of a dissipative function. However, unlike the traditional elasticity, the dissipative function does not have a potential. The fact of an elasticity matrix having asymmetric components means that stress tensor and deformation tensor

are not coaxial. The indicated peculiarities drastically differ the asymmetric elasticity model from the symmetrical one and impose the authors to introduce a new term — "pseudoelasticity" — for it.

The relation (1.1) results in the following inverse dependence

$$\varepsilon = D \sigma, \quad D = C^{-1}. \quad (1.2)$$

Compliance matrix D has the same structure as C . Its zero components are

$$\begin{aligned} D_{14} = D_{15} = D_{24} = D_{25} = D_{34} = D_{35} = D_{36} = D_{41} = D_{42} = \\ D_{43} = D_{46} = D_{51} = D_{52} = D_{53} = D_{56} = D_{63} = D_{64} = D_{65} = 0. \end{aligned}$$

Therefore relation (1.2) can be expressed in the form of a system

$$\begin{aligned} \varepsilon_1 = D_{11}\sigma_1 + D_{12}\sigma_2 + D_{13}\sigma_3 + D_{16}\sigma_6, \quad \varepsilon_2 = D_{21}\sigma_1 + D_{22}\sigma_2 + D_{23}\sigma_3 + D_{26}\sigma_6, \\ \varepsilon_3 = D_{31}\sigma_1 + D_{32}\sigma_2 + D_{33}\sigma_3, \quad \varepsilon_4 = D_{44}\sigma_4 + D_{45}\sigma_5, \\ \varepsilon_5 = D_{54}\sigma_4 + D_{55}\sigma_5, \quad \varepsilon_6 = D_{61}\sigma_1 + D_{62}\sigma_2 + D_{66}\sigma_6. \end{aligned} \quad (1.3)$$

Dynamic and kinematic equations preserve the traditional form. Within the Cartesian reference system the dynamic equations have the following form

$$\sigma_{1,1} + \sigma_{3,2} + \sigma_{4,3} + F_1 = 0, \quad \sigma_{3,1} + \sigma_{2,2} + \sigma_{5,3} + F_2 = 0, \quad \sigma_{4,1} + \sigma_{5,2} + \sigma_{6,3} + F_3 = 0, \quad (1.4)$$

where F_i means components of volume forces (including inertial forces), index after a comma means a derivative at the corresponding coordinate.

If $w_i(x_1, x_2, x_3)$ are displacements of the body points, then traditional kinematic equations are

$$\varepsilon_1 = w_{1,1}, \quad \varepsilon_2 = w_{2,2}, \quad \varepsilon_3 = w_{1,2} + w_{2,1}, \quad \varepsilon_4 = w_{1,3} + w_{3,1}, \quad \varepsilon_5 = w_{2,3} + w_{3,2}, \quad \varepsilon_6 = w_{3,3}. \quad (1.5)$$

Resolvability of this system in relation to w_i functions is provided for by the traditional equations of strains compatibility [8–10].

Equations (1.3) – (1.5) form a closed set within an 3-dimensional asymmetric model of Cauchy elastic continuum.

Accurate Solutions of 3-Dimensional Problems. The solvability conditions of system (1.5) are fulfilled identically for deformations being in

linear dependence of the coordinates. Similar to the traditional elasticity theory [9], a class of problems having an analytical solutions with a linear field of strains may be distinguished (separated??).

Set of equations (1.3) – (1.5) shall be analyzed with the following assumptions:

1) components of matrices C and D do not depend on the coordinates; 2) components of F_i external forces do not depend on the coordinates and time. According to the first assumption, the object of the analysis is a homogeneous body with an asymmetric elasticity tensor. Because of the indicated restrictions, the following linear functions are the solutions of equations (1.4):

$$\sigma_j = a_j x + b_j y + c_j z + p_j \quad (1.6)$$

(x, y, z are Cartesian coordinates) with coefficients a_j, b_j, c_j, p_j , connected with three equalities

$$c_4 = -a_1 - b_3 - F_1, \quad c_5 = -a_3 - b_2 - F_2, \quad c_6 = -a_4 - b_5 - F_3.$$

As a result of (1.3), the strains shall also be linear functions of coordinates

$$\varepsilon_k = f_k x + g_k y + h_k z + e_k, \quad (1.7)$$

with coefficients e_k, f_k, g_k, h_k , calculated via a_j, b_j, c_j, p_j .

The conditions of strains compatibility identically allow a quadratic dependence of displacements of coordinates [9]:

$$w_i = \frac{1}{2} (\alpha_{i1} x^2 + \alpha_{i2} y^2 + \alpha_{i3} z^2) + \alpha_{i4} xy + \alpha_{i5} xz + \alpha_{i6} yz + \alpha_i x + \beta_i y + \gamma_i z + \delta_i. \quad (1.8)$$

These functions contain 30 unknown constants $\alpha_{ij}, \alpha_i, \beta_i, \gamma_i, \delta_i$. Their number can be reduced if to eliminate the solid body displacements.

Exclusion of displacements and rotations of the body in point (0, 0, 0) by means of conditions

$$w_i = 0, \quad w_{1,3} = 0, \quad w_{2,3} = 0, \quad w_{2,1} - w_{1,2} = 0$$

brings us to equalities

$$\delta_i = 0, \quad \gamma_1 = 0, \quad \gamma_2 = 0, \quad \alpha_2 - \beta_1 = 0,$$

that eliminate six constants. The remaining coefficients of polynomial (1.8) are defined from (1.5) by putting the known functions (1.7) into their left parts and the unknown functions (1.8) into the right parts. Thus, all coefficients of polynomial (1.8) can be defined:

$$\begin{aligned}
\alpha_{11} &= f_1, \alpha_{12} = g_3 - f_2, \alpha_{13} = h_4 - f_6, \\
\alpha_{14} &= g_1, \alpha_{15} = h_1, 2\alpha_{16} = h_3 + g_4 - f_5, \\
\alpha_{21} &= f_3 - g_1, \alpha_{22} = g_2, \alpha_{23} = h_5 - g_6, \\
\alpha_{24} &= f_2, 2\alpha_{25} = f_5 + h_3 - g_4, \alpha_{26} = h_2, \\
\alpha_{31} &= f_4 - h_1, \alpha_{32} = g_5 - h_2, \alpha_{33} = h_6, \\
2\alpha_{34} &= f_5 - h_3 + g_4, \alpha_{35} = f_6, \alpha_{36} = g_6, \\
\alpha_1 &= e_1, 2\alpha_2 = 2\beta_1 = e_3, \beta_2 = e_2, \alpha_3 = e_4, \\
\beta_3 &= e_5, \gamma_1 = \gamma_2 = 0, \gamma_3 = e_6, \delta_i = 0.
\end{aligned}$$

Functions (1.6) – (1.8) give an accurate polynomial solution of the class of 3-dimensional problems on deformation of asymmetrically-elastic bodies under volume and surface loads. Let us analyze two typical boundary value problems from this class.

Circular Shaft Torsion. Let L be the shaft length, R is the cross-section radius. Let us match z axis (of material rotational symmetry) with the shaft axis and locate the origin of coordinates at the midsection, so that $(x, y) \in [0, R]$, $z \in [-\frac{1}{2}L, \frac{1}{2}L]$.

Let us assume $F_3 = F_2 = F_1 = 0$ in equations (1.4) and analyze the stress state

$$\sigma_4 = b_4 y, \sigma_5 = a_5 x, \sigma_6 = \sigma_3 = \sigma_2 = \sigma_1 = 0,$$

which is the solution of homogeneous dynamic equations.

When analyzing the shaft torsion, let us request absence of tangential stress in axial section. This condition is expressed by equalities

$$\sigma_{zr} = \sigma_4 \cos \varphi + \sigma_5 \sin \varphi = b_4 r \sin \varphi \cos \varphi + a_5 r \cos \varphi \sin \varphi = 0.$$

and shall be fulfilled provided that $b_4 = -a_5$.

Thus the stress state

$$\sigma_4 = -a_5 y, \sigma_5 = a_5 x, \sigma_6 = \sigma_3 = \sigma_2 = \sigma_1 = 0 \quad (1.9)$$

satisfies homogeneous equations of equilibrium and homogeneous boundary conditions on the cylindric surface and is the state of the shaft pure torsion.

Calculating the tangential stress in the shaft cross-section, we shall have

$$\sigma_{z\varphi} = \sigma_5 \cos \varphi - \sigma_4 \sin \varphi = a_5 r.$$

The obtained solution is accurate when the shaft is twisted by tangential stress $\sigma_{z\varphi} = \tau r/R$, applied to ends of the shaft. The constant $a_5 = \tau/R$ is defined by means of τ parameter which has meaning of tangential stress at the boundary contour of the shaft.

Strains corresponding to stresses (1.9) take the following values

$$\varepsilon_4 = a_5 (D_{45}x - D_{44}y), \quad \varepsilon_5 = a_5 (D_{44}x + D_{45}y), \quad \varepsilon_6 = \varepsilon_3 = \varepsilon_2 = \varepsilon_1 = 0,$$

displacements are calculated by formulas

$$w_1 = -D_{44}a_5 y z, \quad w_2 = D_{44}a_5 x z, \quad 2w_3 = D_{45}a_5 (x^2 + y^2)$$

in the Cartesian reference system or by formulas

$$w_r = 0, \quad w_\varphi = D_{44}a_5 r z, \quad 2w_3 = D_{45}a_5 r^2 \quad (1.10)$$

in the cylindrical system. The traditional solution results from (1.10) when $D_{45} = 0$ [9]:

$$w_r = 0, \quad w_\varphi = D_{44}a_5 r z, \quad w_3 = 0. \quad (1.11)$$

Product $D_{44}a_5 = \omega$ has the meaning of torsion angle per a unit of the shaft length.

Comparison of formulas (1.10) and (1.11) shows that the solution of the modified problem gives the displacements field being different from the traditional one. It contains axial displacements variable along the radius that generate deplanation of the shaft cross sections (Fig.1).

Biaxial Bending of a Plate. The problem is set for a rectangular plate of constant thickness $2h$. The x and y axes are located within the middle plane of the plate, z axis – along the central normal to it. The coordinates are given in domain $x \in [-a, a]$, $y \in [-b, b]$, $z \in [-h, h]$ ($2a$, $2b$ are dimensions of the plate in middle plane).

The plate is bending by stresses

$$\sigma_1 = c_1 z, \quad \sigma_2 = c_2 z, \quad \sigma_3 = \sigma_4 = \sigma_5 = \sigma_6 = 0.$$

This stresses state satisfy the homogeneous equations of equilibrium and homogeneous force conditions at surfaces $z = \mp h$. In solution of (1.6) we have

$$a_k = b_k = p_k = 0, \quad c_6 = c_5 = c_4 = c_3 = 0.$$

Non-zero deformations

$$\varepsilon_1 = D_{11}\sigma_1 + D_{12}\sigma_2, \quad \varepsilon_2 = D_{21}\sigma_1 + D_{22}\sigma_2, \quad \varepsilon_3 = D_{31}\sigma_1 + D_{32}\sigma_2$$

are linear functions of z coordinate. Displacements are calculated by formulae

$$w_1 = z(\alpha_{15}x + \alpha_{16}y), \quad w_2 = z(\alpha_{25}x + \alpha_{26}y), \quad 2w_3 = \alpha_{31}x^2 + \alpha_{32}y^2 + 2\alpha_{34}xy \quad (1.12)$$

with constant coefficients (in case of given c_1 and c_2):

$$\begin{aligned} \alpha_{15} &= D_{11}c_1 + D_{12}c_2, \quad 2\alpha_{16} = D_{31}c_1 + D_{32}c_2, \quad \alpha_{26} = D_{21}c_1 + D_{22}c_2, \\ \alpha_{25} &= \alpha_{16}, \quad \alpha_{31} = -\alpha_{15}, \quad \alpha_{32} = -\alpha_{26}, \quad \alpha_{34} = -\alpha_{16}. \end{aligned}$$

In case of an uniaxial bend by stress σ_2 ($\sigma_1 = 0$), deformations of the plate sections corresponding to solution of (1.12) are schematically represented in Fig.4-6. Unlike the traditional solution, the plate is curved not only in relation to y axis (Fig.4), but in relation to x axis as well (Fig.5). Fig.6 demonstrates level lines on equidistant surfaces of the plate. We can notice the rotation of asymptotes of hyperbolic lines, which is absent within the traditional solution [9].

2. 2-Dimensional Model of the Asymmetrical Pseudoelasticity

If deformation is parallel to plane xy (plane deformation), so that $\varepsilon_4 = \varepsilon_5 = \varepsilon_6 = 0$, a six-equations system (1.1) degenerates into a three-equations system

$$\sigma = A\varepsilon, \quad A = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{12} & C_{11} & -C_{13} \\ -C_{13} & C_{13} & C_{33} \end{bmatrix}. \quad (2.1)$$

In other words, matrix C degenerates into matrix A of plane deformation problem.

Let us introduce displacements vector $w = (u, v)$ and components of the linear strains tensor

$$\varepsilon_{11} = \varepsilon_1 = u_{,x}, \quad \varepsilon_{22} = \varepsilon_2 = v_{,y}, \quad 2\varepsilon_{12} = \varepsilon_3 = u_{,y} + v_{,x}.$$

The solvability of this system with regard to functions u and v is provided for by a traditional strain compatibility equation [9, 10]

$$\varepsilon_{11,yy} + \varepsilon_{22,xx} - 2\varepsilon_{12,xy} = 0.$$

As long as $C_{12} = C_{11} - 2C_{33}$, matrix A has got only three independent components. Let us introduce three kinetic parameters λ_0, μ_0, μ , so that $C_{11} = \lambda_0 + \mu$, $C_{12} = \lambda_0 - \mu$, $C_{13} = \mu_0$, $C_{33} = \mu$. Then relations (2.1) can be put in a tensor form

$$T = I \lambda_0 \operatorname{div} w + 2M \gamma, \quad (2.2)$$

$$T = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix}, \quad M = \begin{pmatrix} \mu & \mu_0 \\ -\mu_0 & \mu \end{pmatrix}, \quad 2\gamma = \begin{pmatrix} u_{,x} - v_{,y} & u_{,y} + v_{,x} \\ v_{,x} + u_{,y} & v_{,y} - u_{,x} \end{pmatrix},$$

where I is a unit tensor, T is a stress tensor, M is the kinetic parameters tensor, and γ is a strain tensor-deviator. Formula (2.2) reveals physical sense of parameters μ and μ_0 . These are shear modules of an elastic body under the conditions of plane deformation with one admissible rotation operator.

Constitutive equations of type (2.2) were obtained earlier by V.O. Bytev [6] for viscous stresses. As opposed to traditional equations, they contain three kinetic parameters: λ_0, μ_0, μ , where $\lambda_0 > 0$, $\mu > 0$, and μ_0 may have any real value. It is also worth mentioning that relation μ_0/μ defines the measure of noncoaxiality of stress and strain tensors-deviators [6], and the substitution $\lambda_0 = \lambda + \mu$, $\mu_0 = 0$ will turn (2.2) into a traditional system of constitutive equations with parameters λ and μ [9, 10].

Relations opposite to (2.2) are represented in the following way:

$$\begin{aligned} u_{,x} &= \frac{1}{4} [\lambda_0^{-1} (\sigma_{11} + \sigma_{22}) + \mu \kappa_0^{-2} (\sigma_{11} - \sigma_{22}) - 2\mu_0 \kappa_0^{-2} \sigma_{12}], \\ v_{,y} &= \frac{1}{4} [\lambda_0^{-1} (\sigma_{11} + \sigma_{22}) - \mu \kappa_0^{-2} (\sigma_{11} - \sigma_{22}) + 2\mu_0 \kappa_0^{-2} \sigma_{12}], \\ u_{,y} + v_{,x} &= \frac{1}{2} \kappa_0^{-2} [\mu_0 (\sigma_{11} - \sigma_{22}) + 2\mu \sigma_{12}], \end{aligned} \quad (2.3)$$

where $\kappa_0^2 = \mu^2 + \mu_0^2$.

Let us take the homogeneous equations of plane problem statics

$$\sigma_{11,x} + \sigma_{12,y} = 0, \quad \sigma_{12,x} + \sigma_{22,y} = 0$$

and introduce Airy function U and function Q : $Q = \sigma_{11} + \sigma_{22} = \Delta U$, $\sigma_{11} = U_{,yy} = Q - U_{,xx}$, $\sigma_{22} = U_{,xx} = Q - U_{,yy}$, $\sigma_{12} = -U_{,xy}$. the first two equations (2.3) shall be transformed as follows:

$$\begin{aligned} u_{,x} &= \frac{1}{4} (\lambda_0^{-1} + \mu\kappa_0^{-2}) Q - \frac{1}{2} \mu\kappa_0^{-2} U_{,xx} + \frac{1}{2} \mu_0\kappa_0^{-2} U_{,xy}, \\ v_{,y} &= \frac{1}{4} (\lambda_0^{-1} + \mu\kappa_0^{-2}) Q - \frac{1}{2} \mu\kappa_0^{-2} U_{,yy} - \frac{1}{2} \mu_0\kappa_0^{-2} U_{,xy}. \end{aligned} \quad (2.4)$$

Now following the same pattern as in [10] we will find that $\Delta Q = 0$. Therefore we may conclude that Q is a harmonic function. Let us take R as a conjugate harmonic function in relation to Q :

$$\frac{\partial Q}{\partial x} = \frac{\partial R}{\partial y}, \quad \frac{\partial Q}{\partial y} = -\frac{\partial R}{\partial x}$$

and $Q + iR = f(z)$, where f is an analytic function of argument $z = x + iy$.

Now let us introduce an analytic function $\varphi(z) = p + iq = \frac{1}{4} \int f(z) dz$. In compliance to the Cauchy-Riemann conditions the following equations are formed [10]

$$\frac{\partial p}{\partial x} = \frac{\partial q}{\partial y} = \frac{1}{4} Q, \quad \frac{\partial p}{\partial y} = -\frac{\partial q}{\partial x} = \frac{1}{4} R.$$

With the help of them, equations (2.4) are transformed as follows:

$$\begin{aligned} u_{,x} &= (\lambda_0^{-1} + \mu\kappa_0^{-2}) p_{,x} - \frac{1}{2} \mu\kappa_0^{-2} U_{,xx} + \frac{1}{2} \mu_0\kappa_0^{-2} U_{,xy}, \\ v_{,y} &= (\lambda_0^{-1} + \mu\kappa_0^{-2}) q_{,y} - \frac{1}{2} \mu\kappa_0^{-2} U_{,yy} - \frac{1}{2} \mu_0\kappa_0^{-2} U_{,xy}. \end{aligned}$$

The following representations for displacement vector components are received after integration:

$$\begin{aligned} u &= (\lambda_0^{-1} + \mu\kappa_0^{-2}) p - \frac{1}{2} \mu\kappa_0^{-2} U_{,x} + \frac{1}{2} \mu_0\kappa_0^{-2} U_{,y} + f_1(y), \\ v &= (\lambda_0^{-1} + \mu\kappa_0^{-2}) q - \frac{1}{2} \mu\kappa_0^{-2} U_{,y} - \frac{1}{2} \mu_0\kappa_0^{-2} U_{,x} + f_2(x). \end{aligned} \quad (2.5)$$

The last equation in the system (2.3) is transformed as follows:

$$u_{,y} + v_{,x} = \frac{1}{2} \mu_0\kappa_0^{-2} (U_{,yy} - U_{,xx}) - \mu\kappa_0^{-2} U_{,xy}. \quad (2.6)$$

After calculation of corresponding derived functions (2.5) –

$$u_{,y} = (\lambda_0^{-1} + \mu\kappa_0^{-2}) p_{,y} - \frac{1}{2} \mu\kappa_0^{-2} U_{,xy} + \frac{1}{2} \mu_0\kappa_0^{-2} U_{,yy} + f_{1,y},$$

$$v_{,x} = (\lambda_0^{-1} + \mu\kappa_0^{-2}) q_{,x} - \frac{1}{2} \mu\kappa_0^{-2} U_{,xy} - \frac{1}{2} \mu_0\kappa_0^{-2} U_{,xx} + f_{2,x},$$

we get their sum as compared to (2.6):

$$u_{,y} + v_{,x} = \frac{1}{2} \mu_0\kappa_0^{-2} (U_{,yy} - U_{,xx}) - \mu\kappa_0^{-2} U_{,xy} =$$

$$(\lambda_0^{-1} + \mu\kappa_0^{-2}) (p_{,y} + q_{,x}) - \mu\kappa_0^{-2} U_{,xy} + \frac{1}{2} \mu_0\kappa_0^{-2} (U_{,yy} - U_{,xx}) + f_{1,y} + f_{2,x}.$$

With account of relation $q_{,x} = -p_{,y}$ we get the following equations $u_{,y} + v_{,x} = f_{1,y} + f_{2,x} = 0$, which means that

$$f_1(y) = c(-\varepsilon y + \alpha_1), \quad f_2(x) = c(\varepsilon x + \alpha_2), \quad (2.7)$$

where $c, \alpha_1, \alpha_2, \varepsilon$ are constant values.

Now lets us go back to (2.5). Taking into account that additional summands of the type (2.7), define rigid translation eliminated by transfer into a new system of coordinates, we get simpler representations for displacement vector components:

$$\begin{aligned} u &= (\lambda_0^{-1} + \mu\kappa_0^{-2}) p - \frac{1}{2} \mu\kappa_0^{-2} U_{,x} + \frac{1}{2} \mu_0\kappa_0^{-2} U_{,y}, \\ v &= (\lambda_0^{-1} + \mu\kappa_0^{-2}) q - \frac{1}{2} \mu\kappa_0^{-2} U_{,y} - \frac{1}{2} \mu_0\kappa_0^{-2} U_{,x}. \end{aligned} \quad (2.8)$$

where $p = p(x, y)$, $q = q(x, y)$.

After introduction of a complex relations definition (2.8)

$$u + iv = (\lambda_0^{-1} + \mu\kappa_0^{-2}) (p + iq) - \frac{1}{2} \kappa\kappa_0^{-2} (U_{,x} + iU_{,y}),$$

symbol $\kappa = \mu + i\mu_0$ and taking into account that $p + iq = \varphi(z)$, and U is a biharmonic function, we shall get the following complex representation for a displacement vector:

$$2\kappa_0^2 (u + iv) = (2\lambda_0^{-1}\kappa_0^2 + \overline{\kappa}) \varphi(z) - \kappa z \overline{\varphi'(z)} - \kappa \overline{\psi(z)}, \quad (2.9)$$

which is a extension of traditional Love formulas [8, 10] (prime means derivative with respect to z).

As for complex representation of stress components with the help of similar functions ϕ and ψ used for the representation of a biharmonic Airy function, it is no way different from the traditional one [10]:

$$\sigma_{11} + \sigma_{22} = 2 \left[\varphi'(z) + \overline{\varphi'(z)} \right] = 4 \operatorname{Re} [\varphi'(z)], \quad \sigma_{22} - \sigma_{11} + i\sigma_{12} = 2 [\bar{z}\varphi''(z) + \psi''(z)].$$

Boundary condition in the second basic problem of the elasticity theory regarding determination of elastic equilibrium on condition of the given boundary displacements shall be represented as follows:

$$(2\lambda_0^{-1}\kappa_0^2 + \bar{\kappa}) \varphi(z) - \kappa z \overline{\varphi'(z)} - \kappa \overline{\psi(z)} = 2\kappa_0^2 (q_1 + iq_2),$$

where q_1 and q_2 are set displacements of boundary points.

Let us assume that two complex planes Z and G and conformal mapping $z = \omega(\zeta)$ of area $S \subset Z$ to area $\Sigma \subset G$ are given. Now let us introduce polar coordinates (r, θ) at G plane, so that $\omega(\zeta) = \zeta = re^{i\theta}$. Then any vector (w_x, w_y) is transformed in compliance to the following formula: $w_r + iw_\theta = e^{-i\theta}(w_x + iw_y)$. With the help of this we can get polar representation of displacement vector $w = (u_r, u_\theta)$ from (2.9):

$$2\kappa_0^2 [u_r + iu_\theta] = e^{-i\theta} \left((2\lambda_0^{-1}\kappa_0^2 + \bar{\kappa}) \varphi(\zeta) - \kappa \zeta \overline{\varphi'(\zeta)} - \kappa \overline{\psi(\zeta)} \right). \quad (2.10)$$

Let us proceed to the following problems in order to demonstrate new effects of plane deformation of asymmetrically-elastic plates.

Circular Washer under Uniform Pressure. In this problem as in traditional elasticity we see the following [10]:

$$\varphi(z) = -\frac{1}{2}pz, \quad \psi(z) = 0, \quad \sigma_{rr} = -p, \quad \sigma_{\theta\theta} = -p, \quad \sigma_{r\theta} = 0.$$

As opposed to the traditional solution, displacement vector (2.10) has got not only radial but also angular component vanishing when $\mu_0 = 0$:

$$u_r = -\frac{1}{2}\lambda_0^{-1}pr, \quad u_\theta = \frac{1}{2}\mu_0\kappa_0^{-2}pr.$$

Biaxial Tension of a Plate with a Circular Hole. In this case we get the following [10]:

$$\varphi(z) = \frac{1}{2}pz, \quad \psi(z) = -pR^2z^{-1},$$

$$\sigma_{rr} = p \left(1 - \frac{R^2}{r^2} \right), \quad \sigma_{\theta\theta} = p \left(1 + \frac{R^2}{r^2} \right), \quad \sigma_{r\theta} = 0$$

(R is the hole radius). Relevant (2.10) displacement vector components are represented by the formulas

$$u_r = \frac{1}{2} p R \left(\lambda_0^{-1} \frac{r}{R} + \mu \kappa_0^{-2} \frac{R}{r} \right), \quad u_\theta = \frac{1}{2} p R \mu_0 \kappa_0^{-2} \left(\frac{R}{r} - \frac{r}{R} \right), \quad (2.11)$$

and their values at boundary $r = R$ -

$$u_r|_{r=R} = \frac{1}{2} p R (\lambda_0^{-1} + \mu \kappa_0^{-2}), \quad u_\theta|_{r=R} = 0. \quad (2.12)$$

Taking into account that $\kappa_0^2 = \mu^2 + \mu_0^2$ and representing parameter λ_0 in the form of the sum $\lambda_0 = \lambda + \mu$, we pass to the limit in (2.11) and (2.12) with $\mu \rightarrow 0$:

$$u_r \rightarrow \frac{p r}{2\lambda}, \quad u_\theta \rightarrow \frac{p R}{2\mu_0} \left(\frac{R}{r} - \frac{r}{R} \right), \quad u_r|_{r=R} \rightarrow \frac{p R}{2\lambda}, \quad u_\theta|_{r=R} = 0.$$

If $\mu_0 = 0$, formulas (2.11) take a traditional form [10]

$$u_r^{cl} = \frac{1}{2} p R \left(\lambda_0^{-1} \frac{r}{R} + \mu^{-1} \frac{R}{r} \right), \quad u_\theta^{cl} \equiv 0.$$

When $\mu \rightarrow 0$, passage to the limit here does not have any physical sense.

Uniaxial Tension of a Plate with a Circular Hole. Let us assume that the contour of the hole is free from external stress and at infinity $\sigma_{11}^\infty = p$, $\sigma_{22}^\infty = 0$, $\sigma_{12}^\infty = 0$. It means that tension is present along Ox axis, and tensile stress at infinity being a constant value p . In this case functions $\varphi(z)$, $\psi(z)$ and stress tensor components shall be defined with the help of the following equations [10]:

$$\begin{aligned} \varphi(z) &= \frac{1}{4} p R \left(\frac{z}{R} + 2 \frac{R}{z} \right), \quad \psi(z) = -\frac{1}{2} p R \left(\frac{z}{R} + \frac{R}{z} - \frac{R^3}{z^3} \right), \\ \sigma_{rr} &= \frac{p}{2} \left[1 - \frac{R^2}{r^2} + \left(1 - 4 \frac{R^2}{r^2} + 3 \frac{R^4}{r^4} \right) \cos 2\theta \right], \\ \sigma_{\theta\theta} &= \frac{p}{2} \left[1 + \frac{R^2}{r^2} - \left(1 + 3 \frac{R^4}{r^4} \right) \cos 2\theta \right], \quad \sigma_{r\theta} = -\frac{p}{2} \left(1 + 2 \frac{R^2}{r^2} - 3 \frac{R^4}{r^4} \right) \sin 2\theta. \end{aligned}$$

Relevant (2.11) displacement vector components are as follows:

$$u_r = \frac{1}{4} pR \left(\lambda_0^{-1} \frac{r}{R} + \mu \kappa_0^{-2} \frac{R}{r} \right) + \frac{1}{4} \mu_0 \kappa_0^{-2} pR \left(\frac{r}{R} - 2 \frac{R}{r} + \frac{R^3}{r^3} \right) \sin 2\theta +$$

$$\frac{1}{4} pR \left[2\lambda_0^{-1} \frac{R}{r} + \mu \kappa_0^{-2} \left(\frac{r}{R} + 2 \frac{R}{r} - \frac{R^3}{r^3} \right) \right] \cos 2\theta, \quad (2.13)$$

$$u_\theta = \frac{1}{4} \mu_0 \kappa_0^{-2} pR \left(\frac{R}{r} - \frac{r}{R} \right) + \frac{1}{4} \mu_0 \kappa_0^{-2} pR \left(\frac{r}{R} - \frac{R^3}{r^3} \right) \cos 2\theta -$$

$$\frac{1}{4} pR \left[2\lambda_0^{-1} \frac{R}{r} + \mu \kappa_0^{-2} \left(\frac{r}{R} + \frac{R^3}{r^3} \right) \right] \sin 2\theta. \quad (2.14)$$

Assuming that $\mu_0 = 0$, we get traditional formulas of this problem [10]:

$$u_r^{cl} = \frac{1}{4} pR \left(\lambda_0^{-1} \frac{r}{R} + \mu^{-1} \frac{R}{r} \right) + \frac{1}{4} pR \left[2\lambda_0^{-1} \frac{R}{r} + \mu^{-1} \left(\frac{r}{R} + 2 \frac{R}{r} - \frac{R^3}{r^3} \right) \right] \cos 2\theta,$$

$$u_\theta^{cl} = -\frac{1}{4} pR \left[2\lambda_0^{-1} \frac{R}{r} + \mu^{-1} \left(\frac{r}{R} + \frac{R^3}{r^3} \right) \right] \sin 2\theta.$$

Now let us calculate displacement values at the contour $r = R$ with the help of (2.13) and (2.14):

$$u_r|_{r=R} = \frac{pR (\mu_0^2 + 2\mu^2 + \lambda\mu)}{4(\lambda + \mu)(\mu_0^2 + \mu^2)} (1 + 2 \cos 2\theta), \quad u_\theta|_{r=R} = -\frac{pR (\mu_0^2 + 2\mu^2 + \lambda\mu)}{2(\lambda + \mu)(\mu_0^2 + \mu^2)} \sin 2\theta.$$

Passage to the limit with $\mu \rightarrow 0$ will give final values of boundary displacements:

$$u_r|_{r=R} \rightarrow \frac{pR}{4\lambda} (1 + 2 \cos 2\theta), \quad u_\theta|_{r=R} \rightarrow -\frac{pR}{2\lambda} \sin 2\theta.$$

Traditional formulas give the following values

$$u_r^{cl}|_{r=R} = \frac{pR (\lambda + 2\mu)}{4\mu (\lambda + \mu)} (1 + 2 \cos 2\theta), \quad u_\theta^{cl}|_{r=R} = -\frac{pR (\lambda + 2\mu)}{2\mu (\lambda + \mu)} \sin 2\theta,$$

having no physical sense when $\mu \rightarrow 0$.

Conclusion. A non-traditional version of the elasticity theory suggested herein contains additional kinetic parameters and requires special experiments for their evaluation. Thus, it provides a scientific basis and new opportunities for experimental analysis. The authors hope that the abandonment of a traditional condition of elasticity tensor symmetry shall considerably expand the opportunities of both linear and nonlinear elasticity theories. Asymmetric non-polar medium theory can be applicable to modeling of anomalies connected to media and materials microstructure.

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